

# Modeling the Eberhard inequality based tests

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## Abstract

Last year the first experimental tests closing the detection loophole (also referred to as the fair sampling loophole) were performed by two experimental groups [1], [2]. To violate Bell-type inequalities (the Eberhard inequality in the first test and the Clauser-Horne inequality in the second test), one has to optimize a number of parameters involved in the experiment (angles of polarization beam splitters and quantum state parameters). Although these are technicalities, their optimal determination plays an important role in approaching statistically significant violations of the inequalities. In this paper we study this problem for the Eberhard inequality in very detail by using the advanced method of numerical optimization, the Nelder-Mead method. First of all, we improve the results of optimization for the original Eberhard model [3] and the Gustina et al. work [1] (“Vienna-13 experiment”) by using the model of this experiment presented in Kofler et al. [4]. We also take into account the well known fact that detectors can have different efficiencies and perform the corresponding optimization. In previous studies the objective function had the meaning of the mathematical expectation. However, it is also useful to investigate the possible level of variability of the results, expressed in terms of standard deviation. In this paper we consider the optimization of parameters for the Eberhard inequality using coefficient of variation taking into account possible random fluctuations in the setup of angles during the experiment.

# 1 Introduction

Experimental realization of a loophole-free test for Bell [5] inequalities will play a crucial role both for quantum foundations [5], [6]–[8], [11]–[13], [9], [10] (see, e.g., [14]–[20] for recent studies) and quantum technologies, e.g., quantum cryptography and quantum random generators. It is clear that the often used argument that “closing of different loopholes in separate tests can be considered as the solution of the loopholes problem” can not be considered as acceptable. The quantum community put tremendous efforts to perform a loophole-free test and its final realization (which can be expected rather soon) will be a great event in development of quantum theory and quantum technologies.

Last year the first experimental tests for photons closing the *detection loophole* (also referred to as the fair sampling loophole) were performed by two experimental groups [1], [2], see also [21]–[23]. The detection efficiency problem for photons is very complicated and its solution was based on the use of advanced photo-detectors, i.e., new technology as well as its testing [24]. The Bell tests with photons [11]–[13] are promising to close both the detection and locality loopholes, since the latter was closed long ago [25] and recently experiments demonstrating violation of Bell-type inequalities on large distances [26]–[33] were performed. However, to violate Bell type inequalities one has to approach very high efficiency of the total experimental setup. Hence, although nowadays it is possible to work with photo-detectors having the efficiency approaching 100%, the losses in the total experimental setup can decrease essentially the total efficiency of the experimental scheme, see [34]–[38] for theoretical analysis and mathematical modeling. The experimentalists confront this problem by trying to extend Bell-type tests with sufficiently high efficiency to close the locality loophole. The total efficiency of experimental schemes decreases drastically with the distance. Therefore it is important to optimize all parameters of the experiment to approach the maximal violation for the minimal possible efficiency. (One has to optimize angles of polarization beam splitters and the initial state parameters). Although these are technicalities, they optimal determination play an important role in approaching statistically significant violations of the inequalities.

In this paper we study this problem for the Eberhard inequality [3] in very detail by using the advanced method of numerical optimization, the *Nelder-Mead method* [39]. First of all, we improve the the results of optimization for the original Eberhard model [3] and the Gustina et al. work [1] (“Vienna-13 experiment”) by using the model of this experiment presented in Kofler et al. [4]. We also take into account the well known fact that detectors can have different efficiencies and perform the corresponding optimization.

In the previous studies [3], [1], [4] the objective function for parameters optimization had the meaning of the mathematical expectation. However, it is also

useful to investigate the possible level of variability of the results, expressed in terms of standard deviation. In this paper we consider the optimization of parameters for the Eberhard inequality using the coefficient  $K$  –the reciprocal of the *coefficient of variation* taking into account possible *random fluctuations in the setup of angles during the experiment*.<sup>1</sup> It seems that our study is the first contribution to this problem. This study (of the problem of sensitivity of the degree of violation of the Eberhard inequality to the precision in the control of angles of polarization beam splitters) can be useful for experimentalists. One of the results of our numerical simulation is unexpected stability of the degree of violation of the Eberhard inequality to fluctuations of these angles (in neighborhoods of optimal values of the angles).

## 2 Eberhard inequality

We follow Eberhard [3]: Photons are emitted in pairs  $(a, b)$ . Under each measurement setting  $(\alpha, \beta)$ , the events in which the photon  $a$  is detected in the ordinary and extraordinary beams are denoted by the symbols  $(o)$  and  $(e)$ , respectively, and the event that it is undetected is denoted by the symbol  $(u)$ . The same symbols are used to denote the corresponding events for the photon  $b$ . Therefore for the pairs of photons there are nine types of events:  $(o, o), (o, u), (o, e), (u, o), (u, u), (u, e), (e, o), (e, u)$ , and  $(e, e)$ .

Under the conditions of locality, realism and statistical reproducibility the following inequality (the Eberhard inequality) was derived:

$$J \equiv n_{oe}(\alpha_1, \beta_2) + n_{ou}(\alpha_1, \beta_2) + n_{eo}(\alpha_2, \beta_1) + n_{uo}(\alpha_2, \beta_1) + n_{oo}(\alpha_2, \beta_2) - n_{oo}(\alpha_1, \beta_1) \geq 0, \quad (1)$$

where  $n_{xy}(\alpha_i, \beta_j)$  is the number of pairs detected in a given time period for settings  $\alpha_i, \beta_j$  with outcomes  $x, y = o, e, u$  and the outcomes  $(o)$  and  $(e)$  correspond to detections in the ordinary and extraordinary beams, respectively, and the event that photon is undetected is denoted by the symbol  $(u)$ . We point to the main distinguishing features of the E-inequality:

- a) derivation without the fair sampling assumption (and without the no-enhancement assumption);
- b) taking into account undetected photons;
- c) background events are taken into account;
- d) the *linear form of presentation* (non-negativity of a linear combination of coincidence and single rates).

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<sup>1</sup>In classical signal analysis this quantity is known as signal/noise rate [40], [41].

The latter feature (which is typically not emphasized in the literature) is crucial to find a simple procedure of optimization of experimental parameters and, hence, it makes the E-inequality the most promising experimental test to close the detection loophole and to reject local realism without the fair sampling assumption. Eberhard's optimization has two main outputs which play an important role in the experimental design:

E1). It is possible to perform an experiment without fair sampling assumption for detection efficiency less than 82,8%. Nevertheless, detection efficiency must still be very high, at least 66.6% (in the absence of background).

E2). The optimal parameters correspond to non-maximally entangled states.

In 2013, the possibility to proceed with overall efficiencies lower than 82.8% (but larger than 66.6%) was explored for the E-inequality and the first experimental test ("the Vienna test") closing the detection loophole was published [1], for more detailed presentation of statistical data see also [21], [4].

## 2.1 Eberhard inequality and quantum mechanical probabilities

Since the use of the Eberhard inequality is not common in quantum foundational studies, we present here in details calculation of quantum mechanical probabilities which violate it (for specially selected parameters of the experimental test). Here we follow the original paper of Eberhard, but we try to adapt the presentation for our purpose of improvement of optimization of parameters. Consider two detectors with the same efficiency  $\eta$ , which perform measurements in  $N$  experiments. That is, in every experiment each of the detectors detects a photon in one of the trajectories with probability  $\eta$ . Let us construct density operators for particles in the ordinary and extraordinary beams. We will use the helicity basis for derivation of this operators.

Two main circular polarization states are described with the following vectors:  $u = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$ ,  $v = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$ . They form the transformation matrix from standard basis to helicity basis:  $W = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$ . The inverse transformation can be made with the inverse matrix:  $T = W^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -1 \end{pmatrix}$ . Then, consider a polarization prism which is rotated by an angle  $\theta$ . A particle which appeared in the ordinary beam has the following state:  $\psi_o = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$  and a particle in the extraordinary beam appeared with the state:  $\psi_e = \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}$ . In the helicity basis this states

are described by the following equations, respectively:  $\psi'_o = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\theta} \\ e^{-i\theta} \end{pmatrix}$ ,  $\psi'_e = \frac{1}{\sqrt{2}} \begin{pmatrix} ie^{i\theta} \\ -ie^{-i\theta} \end{pmatrix}$ . Then, corresponding density operators have the following form:  $P_o = \psi'_o \cdot \psi'^{\dagger}_o = \frac{1}{2} \begin{pmatrix} 1 & e^{2i\theta} \\ e^{-2i\theta} & 1 \end{pmatrix}$ ,  $P_e = \psi'_e \cdot \psi'^{\dagger}_e = \frac{1}{2} \begin{pmatrix} 1 & -e^{2i\theta} \\ -e^{-2i\theta} & 1 \end{pmatrix}$ . Density operators of the considering system with two particles are described by tensor products. If the first particle appeared in the ordinary beam, it is described by  $P_o \otimes I$ , if the second particle appeared in the ordinary beam, then the corresponding operator is given by  $I \otimes P_o$ . Operators for particles in extraordinary beam can be constructed in the similar way. Finally, putting it all together and using the formula of quantum expectation value, for the initial state of the system  $\psi$  quantum mechanics predicts the following results:

$$n_{oo}(\alpha_1, \beta_1) = N \frac{\eta^2}{4} \psi^\dagger [I + \sigma(\alpha_1)] [I + \tau(\beta_1)] \psi, \quad (2)$$

$$n_{oe}(\alpha_1, \beta_2) = N \frac{\eta^2}{4} \psi^\dagger [I + \sigma(\alpha_1)] [I - \tau(\beta_2)] \psi, \quad (3)$$

$$n_{ou}(\alpha_1, \beta_2) = N [\eta(1 - \eta)/2] \psi^\dagger [I + \sigma(\alpha_1)] \psi, \quad (4)$$

$$n_{eo}(\alpha_2, \beta_1) = N \frac{\eta^2}{4} \psi^\dagger [I - \sigma(\alpha_2)] [I + \tau(\beta_1)] \psi, \quad (5)$$

$$n_{uo}(\alpha_2, \beta_1) = N [\eta(1 - \eta)/2] \psi^\dagger [I + \tau(\beta_1)] \psi, \quad (6)$$

$$n_{oo}(\alpha_2, \beta_2) = N \frac{\eta^2}{4} \psi^\dagger [I + \sigma(\alpha_2)] [I + \tau(\beta_2)] \psi, \quad (7)$$

where

$$\sigma(\alpha) = \begin{vmatrix} 0 & e^{2i(\alpha-\alpha_1)} & 0 & 0 \\ e^{-2i(\alpha-\alpha_1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{2i(\alpha-\alpha_1)} \\ 0 & 0 & e^{-2i(\alpha-\alpha_1)} & 0 \end{vmatrix}$$

and

$$\tau(\beta) = \begin{vmatrix} 0 & 0 & e^{2i(\beta-\beta_1)} & 0 \\ 0 & 0 & 0 & e^{2i(\beta-\beta_1)} \\ e^{-2i(\beta-\beta_1)} & 0 & 0 & 0 \\ 0 & e^{-2i(\beta-\beta_1)} & 0 & 0 \end{vmatrix}.$$

Note that matrices  $I + \sigma(\alpha)$ ,  $I - \sigma(\alpha)$ ,  $I + \tau(\beta)$  and  $I - \tau(\beta)$  can be simply derived from previously considered density operators with the properly choosen value of  $\theta$ . Also note that probability to fail to detect particle equals to  $1 - \eta$  no matter what particle is considered, first of second.

Thus, we obtain the Eberhard inequality for quantum mechanical quantities:

$$J_{\mathcal{B}}^{\text{ideal}} = n_{uo}(\alpha_2, \beta_1) + n_{eo}(\alpha_2, \beta_1) + n_{ou}(\alpha_1, \beta_2) + n_{oe}(\alpha_1, \beta_2) + n_{oo}(\alpha_2, \beta_2) - n_{oo}(\alpha_1, \beta_1) \geq 0.$$

However, in reality, in addition to correct detections during experiments some false positives may arise that is called background. In the Eberhard model it is assumed that the number of false positive detections for events of type  $(o, o)$  can be ignored. We assume that the level of background does not depend on  $\alpha$  and  $\beta$ , so for events  $n_{uo}(\alpha_2, \beta_1) + n_{eo}(\alpha_2, \beta_1)$  and  $n_{ou}(\alpha_1, \beta_2) + n_{oe}(\alpha_1, \beta_2)$ , it has the same value  $N\zeta$ . The resulting inequality takes the form:  $J_{\mathcal{B}} = J_{\mathcal{B}}^{\text{ideal}} + 2N\zeta \geq 0$ .

This inequality can be written as  $\psi^\dagger \mathcal{B} \psi \geq 0$ , where  $\mathcal{B}$  is a matrix:

$$\mathcal{B} = N \frac{\eta}{2} \begin{pmatrix} 2 - \eta + \xi & 1 - \eta & 1 - \eta & A^* B^* - \eta \\ 1 - \eta & 2 - \eta + \xi & AB^* - \eta & 1 - \eta \\ 1 - \eta & A^* B - \eta & 2 - \eta + \xi & 1 - \eta \\ AB - \eta & 1 - \eta & 1 - \eta & 2 - \eta + \xi \end{pmatrix},$$

where  $A = \eta/2(e^{2i(\alpha_1 - \alpha_2)} - 1)$ ,  $B = e^{2i(\beta_1 - \beta_2)} - 1$ ,  $\xi = 4\zeta/\eta$ .

For the implementation of the experiment, which could show a violation of this inequality, we will search for the parameters such that  $J_{\mathcal{B}} < 0$ . Consider the case when  $\zeta = 0$ , that is, the detectors do not give false positives, and  $\alpha_1 - \alpha_2 =$

$$\beta_1 - \beta_2 = \theta. \text{ We will use the following quantum state: } \psi = \frac{1}{2\sqrt{1+r^2}} \begin{pmatrix} (1+r)e^{-i\omega} \\ -(1-r) \\ -(1-r) \\ (1+r)e^{i\omega} \end{pmatrix},$$

where  $0 \leq r \leq 1$ ,  $\alpha_1 = \omega/2 - 90^\circ$  and  $\beta_1 = \omega/2$ .

### 3 The results of optimization of parameters for experimental tests based on Eberhard's inequality

Our numerical optimization of parameters of the experimental tests to violate Eberhard's inequality will be based on *the Nelder-Mead optimization method*. The Nelder-Mead method [39] (also known as downhill simplex method) is widely used for nonlinear optimization problems. This numerical method is typically applied to problems for which derivatives may not be known. Its applications are especially successful in the case of multi-dimensional spaces of parameters.

In this section we present a part of the results of our studies, the results of numerical optimization of parameters to violate Eberhard's inequality as much as possible. We start with comparison with the original Eberhard model [3], then we

consider the case of detectors having different efficiencies, so in general  $\eta_1 \neq \eta_2$ . Finally, we consider the model, see Kofler et al. [4] which was used in the recent Bell test [1] based on the Eberhard inequality, the “Vienna-13 experiment”

### 3.1 Optimization of parameters for the Eberhard model

For every  $\eta$  let us find parameters  $r, \omega, \theta$  that allows the inequality to be violated most strongly. To do so, we will minimize the function  $f = J_{\mathcal{B}}(r, \omega, \theta)/N$  using the Nelder-Mead method.

$\eta$	$r$	$\omega, ^\circ$	$\theta, ^\circ$	$J_{\mathcal{B}}/N$
0.7	0.136389	3.40081	21.4266	-0.000453562
0.75	0.310518	9.73143	31.9603	-0.00615095
0.8	0.465228	14.8979	37.9215	-0.02191
0.85	0.607424	18.5808	41.5341	-0.0496902
0.9	0.741202	20.9153	43.6381	-0.0899078
0.95	0.87067	22.141	44.6958	-0.142436
1	0.999997	22.5	45	-0.207107

Table 1: Optimal parameters values for  $J_{\mathcal{B}}/N$  from the Eberhard inequality

The values obtained while optimizing  $J_{\mathcal{B}}(r, \omega, \theta)/N$  are shown in Table 1. During the optimization process the  $\zeta$ -values were set to zero, because this parameter brings a constant contribution  $2\zeta$  into the  $J_{\mathcal{B}}/N$  value. It increases the  $J_{\mathcal{B}}/N$  value by a constant regardless of other parameters, therefore not affecting optimization result. Thus values from Table 1 match with values obtained by Eberhard [3] for non-zero context level, and parameters such that the inequality is violated the most strongly for  $\zeta = 0$  match with parameters such that the inequality is violated and  $\zeta$  has the maximum value.

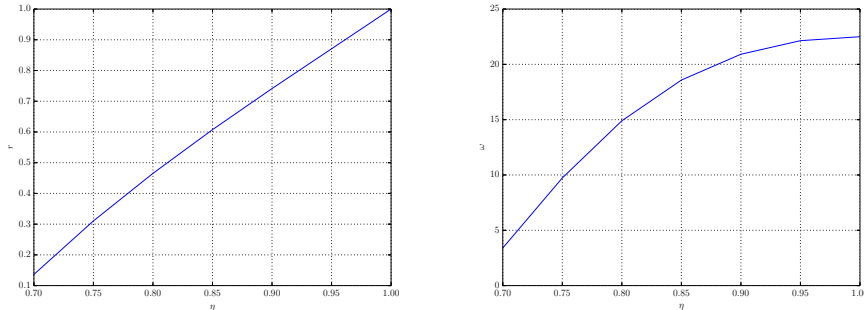


Figure 1: Optimized  $r$  and  $\omega$  values for different detectors efficiency values

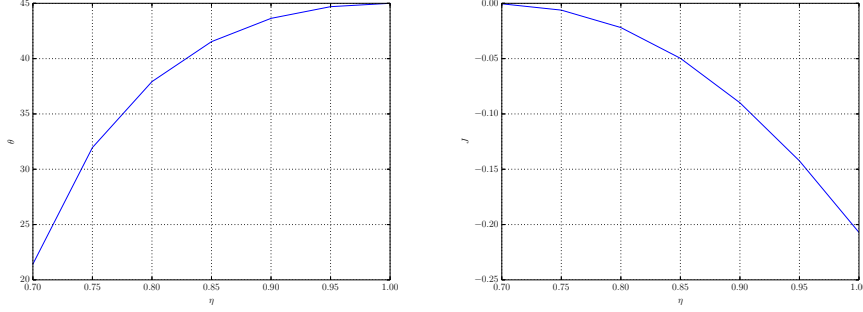


Figure 2: Optimized  $\theta$  qnd  $J_{\mathcal{B}}/N$  values for different detector efficiency values

At Fig. 1 relations between obtained parameters  $r$  and  $\omega$  and the efficiency  $\eta$  are shown, at the first picture of Fig. 2 relations between obtained parameter  $\theta$  and efficiency is shown, at the second picture of this figure dependence of the minimum function values on efficiency is presented.

### 3.2 Optimization of parameters for detectors with different efficiencies; Eberhard model

In real experiments detectors more often have different efficiency values, the formulas (2)-(7) can be easily adapted to this case. The Eberhard inequality can be written as  $\psi^\dagger \mathcal{B} \psi \geq 0$ , where  $\mathcal{B}$  is a matrix:

$$\mathcal{B} = \frac{N}{2} \begin{pmatrix} C + \xi & \eta_1(1 - \eta_2) & \eta_2(1 - \eta_1) & A^*B^* - \eta_1\eta_2 \\ \eta_1(1 - \eta_2) & C + \xi & AB^* - \eta_1\eta_2 & \eta_2(1 - \eta_1) \\ \eta_2(1 - \eta_1) & A^*B - \eta_1\eta_2 & C + \xi & \eta_1(1 - \eta_2) \\ AB - \eta_1\eta_2 & \eta_2(1 - \eta_1) & \eta_1(1 - \eta_2) & C + \xi \end{pmatrix},$$

where  $A = \eta_1/2(e^{2i(\alpha_1 - \alpha_2)} - 1)$ ,  $B = \eta_2/2(e^{2i(\beta_1 - \beta_2)} - 1)$ ,  $C = \eta_1 + \eta_2 - \eta_1\eta_2$  and  $\xi = 4\zeta$ .

At Fig.3 - 4 optimal parameters are shown, together with the minimized function for different detector efficiencies.

$\eta_1$	$\eta_2$	$r$	$\omega, ^\circ$	$\theta, ^\circ$	$J_{\mathcal{B}}/N$	$\zeta$
0.65	0.65	9.73816e-05	5.32909e-05	0.609422	4.13366e-10	—
0.65	0.7	0.0325726	0.4367	10.3918	-5.37452e-06	2.68726e-06
0.65	0.75	0.122804	2.94407	20.3249	-0.000327839	0.00016392
0.65	0.8	0.199412	5.64399	25.8864	-0.00152446	0.000762231
0.65	0.85	0.266448	8.1196	29.7694	-0.00385363	0.00192682
0.65	0.9	0.326193	10.2951	32.6807	-0.00737942	0.00368971



$\eta_1$	$\eta_2$	$r$	$\omega,^\circ$	$\theta,^\circ$	$J_{\mathcal{B}}/N$	$\zeta$
0.65	0.95	0.380046	12.1725	34.9409	-0.0120653	0.00603265
0.65	1	0.42905	13.7822	36.7384	-0.0178271	0.00891353
0.7	0.65	0.0325721	0.436691	10.3917	-5.37452e-06	2.68726e-06
0.7	0.7	0.136389	3.40081	21.4266	-0.000453562	0.000226781
0.7	0.75	0.223629	6.53572	27.3754	-0.00217282	0.00108641
0.7	0.8	0.299639	9.33741	31.4432	-0.00551773	0.00275887
0.7	0.85	0.367155	11.7325	34.4284	-0.0105412	0.00527059
0.7	0.9	0.427895	13.7454	36.6985	-0.0171516	0.0085758
0.7	0.95	0.48304	15.4224	38.4617	-0.0251977	0.0125989
0.7	1	0.533433	16.8118	39.8493	-0.034513	0.0172565
0.75	0.65	0.122804	2.94407	20.3249	-0.000327839	0.00016392
0.75	0.7	0.223629	6.53572	27.3754	-0.00217282	0.00108641
0.75	0.75	0.310518	9.73143	31.9603	-0.00615095	0.00307547
0.75	0.8	0.387235	12.4151	35.2194	-0.0123604	0.0061802
0.75	0.85	0.455977	14.6188	37.6299	-0.0206635	0.0103318
0.75	0.9	0.518149	16.4057	39.45	-0.0308332	0.0154166
0.75	0.95	0.574864	17.8435	40.8423	-0.0426257	0.0213128
0.75	1	0.626962	18.9919	41.9138	-0.0558135	0.0279068
0.8	0.65	0.199412	5.64399	25.8864	-0.00152446	0.000762231
0.8	0.7	0.299639	9.33741	31.4432	-0.00551773	0.00275887
0.8	0.75	0.387235	12.4151	35.2194	-0.0123604	0.0061802
0.8	0.8	0.465228	14.8979	37.9215	-0.02191	0.010955
0.8	0.85	0.535462	16.8648	39.9011	-0.0338613	0.0169307
0.8	0.9	0.59933	18.4036	41.3692	-0.0478775	0.0239387
0.8	0.95	0.657889	19.5935	42.4623	-0.063646	0.031823
0.8	1	0.712001	20.5014	43.274	-0.0808982	0.0404491
0.85	0.65	0.266448	8.1196	29.7694	-0.00385363	0.00192682
0.85	0.7	0.367155	11.7325	34.4284	-0.0105412	0.00527059
0.85	0.75	0.455977	14.6188	37.6299	-0.0206635	0.0103318
0.85	0.8	0.535462	16.8648	39.9011	-0.0338613	0.0169307
0.85	0.85	0.607424	18.5808	41.5341	-0.0496902	0.0248451
0.85	0.9	0.673214	19.8694	42.7109	-0.0677295	0.0338647
0.85	0.95	0.733924	20.817	43.552	-0.087619	0.0438095
0.85	1	0.790464	21.4958	44.1427	-0.109064	0.0545318
0.9	0.65	0.326193	10.2951	32.6807	-0.00737942	0.00368971
0.9	0.7	0.427895	13.7454	36.6985	-0.0171516	0.0085758
0.9	0.75	0.518149	16.4057	39.45	-0.0308332	0.0154166
0.9	0.8	0.59933	18.4036	41.3692	-0.0478775	0.0239387
0.9	0.85	0.673214	19.8694	42.7109	-0.0677295	0.0338647
0.9	0.9	0.741202	20.9153	43.6381	-0.0899078	0.0449539
0.9	0.95	0.804477	21.6349	44.2627	-0.11402	0.0570101

$\eta_1$	$\eta_2$	$r$	$\omega, ^\circ$	$\theta, ^\circ$	$J_{\mathcal{B}}/N$	$\zeta$
0.9	1	0.863896	22.1002	44.661	-0.139755	0.0698776
0.95	0.65	0.380046	12.1725	34.9409	-0.0120653	0.00603265
0.95	0.7	0.48304	15.4224	38.4617	-0.0251977	0.0125989
0.95	0.75	0.574864	17.8435	40.8423	-0.0426257	0.0213128
0.95	0.8	0.657889	19.5935	42.4623	-0.063646	0.031823
0.95	0.85	0.733924	20.817	43.552	-0.087619	0.0438095
0.95	0.9	0.804477	21.6349	44.2627	-0.11402	0.0570101
0.95	0.95	0.87067	22.141	44.6958	-0.142436	0.0712182
0.95	1	0.933431	22.4101	44.924	-0.172546	0.086273
1	0.65	0.42905	13.7822	36.7384	-0.0178271	0.00891353
1	0.7	0.533433	16.8118	39.8493	-0.034513	0.0172565
1	0.75	0.626962	18.9919	41.9138	-0.0558135	0.0279068
1	0.8	0.712001	20.5014	43.274	-0.0808982	0.0404491
1	0.85	0.790464	21.4958	44.1427	-0.109064	0.0545318
1	0.9	0.863896	22.1002	44.661	-0.139755	0.0698776
1	0.95	0.933431	22.4101	44.924	-0.172546	0.086273
1	1	0.999997	22.5	45	-0.207107	0.103553

Table 2: Optimal parameters values for the case with different detector efficiencies

Parameters values obtained during optimization process along with maximal allowable noise level are shown in Table 2. The bigger each value of efficiency separately, the more strongly inequality can be violated. Minimal efficiency values such that the violation is possible are close to  $\eta_1 = \eta_2 = 0.67$  value, matching Eberhard results.

Generally, for every state  $\psi$  that minimize an expectation value the following corollary holds.

**Theorem.** *Quantum state  $\psi$  minimizing target function  $J$  is an eigenvector of the matrix  $\mathcal{B}$  and the dispersion for it is equal to zero.*

This theorem can be proved using Courant-Fischer theorem.

**Theorem**[Courant-Fisher] *Let  $A$  be a  $n \times n$  Hermitian matrix with eigenvalues  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ . RayleighRitz quotient for this matrix is defined by*

$$R_A(x) = \frac{(Ax, x)}{(x, x)}.$$

*For  $1 \leq k \leq n$ , let  $S_k$  denote the span of  $v_1, \dots, v_k$  and let  $S_k^\perp$  denote the orthogonal complement of  $S_k$ . Then*

$$\lambda_1 \leq R_A(x) \leq \lambda_n, \quad \forall x \in \mathbb{C}^n \setminus \{0\}$$

and

$$\lambda_k = \max\{\min\{R_A(x) \mid x \neq 0 \in U\} \mid \dim(U) = k\},$$

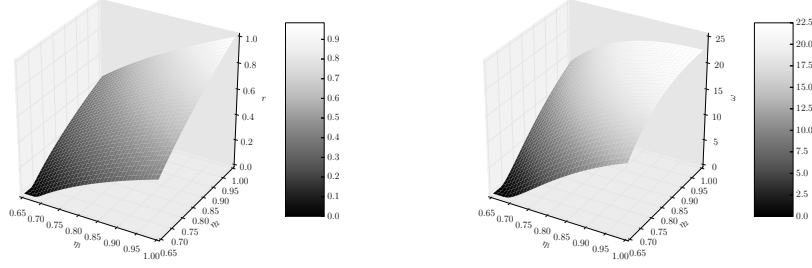


Figure 3: Optimized  $r$  and  $\omega$  values for different detector efficiencies

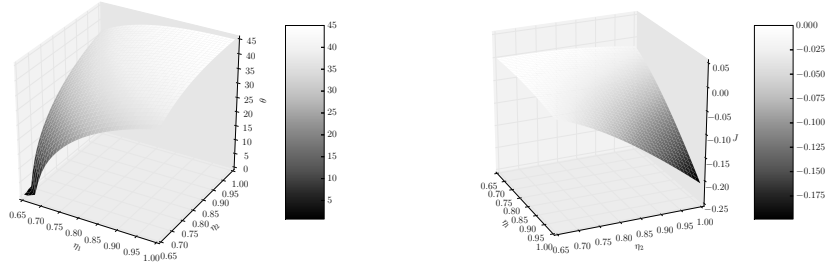


Figure 4: Optimized values of  $\theta$  and  $J_{\mathcal{B}}/N$  for different detector efficiencies

$$\lambda_k = \min\{\max\{R_A(x) \mid x \neq 0 \in U\} \mid \dim(U) = n - k + 1\}$$

It means that the obtained states are optimal not only from the mathematical expectation point of view, but also from possible spread of measurement results point of view expressed in terms of dispersion.

### 3.3 Optimization of parameters in the model for the Vienna-13 experiment

To match the real experimental situation, see Gustina et al. [1] (Vienna-13 experiment), in article of Kofler et al. [4] analysis of the use of the Eberhard inequality in this concrete experiment was performed. This analysis led to the conclusion that data produced in the Vienna-13 experiment [1] is described by a more complicated model (in the standard quantum framework)<sup>2</sup> than the original Eberhard

<sup>2</sup>As one of the aims, the work of Kofler et al. [4] has justification of the statistical output of the Vienna-13 experiment by using standard quantum mechanical tools. This was questioned by the author of the paper [?].

model [3]. This does not decrease the value of the original Eberhard study. Kofler et al. [4] just pointed that some important additional “technicalities” have to be taken into account.

During experiments [1] the values of the quantities  $n_{ou}, n_{uo}$  were found by the following formulas:

$$n_{ou}(\alpha_1, \beta_2) = S_o^A(\alpha_1) - n_{oo}(\alpha_1, \beta_2) - n_{oe}(\alpha_1, \beta_2),$$

$$n_{uo}(\alpha_2, \beta_1) = S_o^B(\beta_1) - n_{oo}(\alpha_2, \beta_1) - n_{eo}(\alpha_2, \beta_1),$$

where  $S_o^A, S_o^B$  – an amount of clicks in the ordinary beam for the first and second system correspondingly.

In this case the Eberhard inequality takes form:

$$J = -n_{oo}(\alpha_1, \beta_1) + S_o^A(\alpha_1) - n_{oo}(\alpha_1, \beta_2) + S_o^B(\beta_1) - n_{oo}(\alpha_2, \beta_1) + n_{oo}(\alpha_2, \beta_2) \geq 0 \quad (8)$$

To model the output of the Vienna-13 experiment, one cannot proceed, as Eberhard did, with pure states. Consider a density operator as a quantum state of the system:

$$\rho = \frac{1}{\sqrt{1+r^2}} \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & Vr & 0 \\ 0 & Vr & r^2 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix},$$

where  $0 \leq r \leq 1$  and  $0 \leq V \leq 1$ . In this case predictions of quantum mechanics for values included into the inequality become:

$$\begin{aligned} \tilde{S}_o^A(\alpha_i) &= \eta_1 N \text{Tr}[\rho(\hat{P}_A(\alpha_i) \otimes I)] \\ \tilde{S}_o^B(\beta_i) &= \eta_1 N \text{Tr}[\rho(I \otimes \hat{P}_B(\beta_i))] \\ \tilde{n}_{oo}(\alpha_i, \beta_i) &= \eta_1 \eta_2 N \text{Tr}[\rho(\hat{P}_A(\alpha_i) \otimes \hat{P}_B(\beta_i))], \end{aligned}$$

where  $\hat{P}_A, \hat{P}_B$  – projection operators on ordinary beam direction for the first and second prisms:

$$\hat{P}(\gamma) = \begin{pmatrix} \cos^2 \gamma & \cos \gamma \sin \gamma \\ \cos \gamma \sin \gamma & \sin^2 \gamma \end{pmatrix}.$$

With regard to false clicks during time  $T$   $S_o^A, S_o^B$  values take the following form:

$$S_o^A(\alpha_i) = \tilde{S}_o^A(\alpha_i) + \zeta T,$$

$$S_o^B(\beta_i) = \tilde{S}_o^B(\beta_i) + \zeta T.$$

Besides noise, the model of the Vienna-13 experiment, see Kofler et al. [4], also considers inconsistencies in time when pairs from different launches are detected as a conjugate events. Let us introduce temporary window value  $\tau_c$ , within

	$\alpha_1, ^\circ$	$\alpha_2, ^\circ$	$\beta_1, ^\circ$	$\beta_2, ^\circ$	$J$
Results from the article	85.6	118.0	-5.4	25.9	-120191
Results of the optimization	85.0	115.1	-4.0	27.4	-126060

Table 3: Optimal parameters comparison with ones from Zeilinger article [1]

which conjugate events must be detected. In this case  $n_{oo}(\alpha_i, \beta_i)$  can be found using the following formulas:

$$n_{oo}(\alpha_i, \beta_i) = \tilde{n}_{oo}(\alpha_i, \beta_i) + n_{oo}^{acc}(\alpha_i, \beta_i),$$

$$n_{oo}^{acc}(\alpha_i, \beta_i) = S_o^A(\alpha_i) S_o^B(\beta_i) \frac{\tau_c}{T} \left( 1 - \frac{\tilde{n}_{oo}(\alpha_i, \beta_i)}{S_o^A(\alpha_i)} \right) \left( 1 - \frac{\tilde{n}_{oo}(\alpha_i, \beta_i)}{S_o^B(\beta_i)} \right).$$

For this model, optimization for the quantity  $J$  given by the expression (8) was performed, and, for the selected values of experimental parameters ( $\eta_1, \eta_2, r, V, T, \tau_c, N, \zeta$ ), the values of the angles that minimize the target function were found. In particular, we remark that Gustina et al. [1] approached the following levels of detectors efficiencies:  $\eta_1 = 73.77$  and  $\eta_2 = 78.59$ .

Optimization results are shown in Table 3. According to obtained values, optimal angle values for prism installation differ from given in [1] and let inequality to be violated stronger. We also point out that asymmetry in detectors' contributions leads to a possibility to play with this asymmetry. In particular, we found that if experimentalists who did the Vienna-13 experiment were simply permuted the detectors, they would get a stronger violation:  $J = -123050$ .

## 4 Optimization of parameters for randomly fluctuating angles of polarization beam splitters

In the Eberhard model [3] and the model for the Vienna-13 experiment [1] optimization of experimental parameters was performed under the assumption that the angles of polarization beam splitters can be chosen exactly. The optimization gives some concrete values and it was assumed that experimentalists can setup the experimental design with precisely these angles. However, this assumption does not match the real experimental situation. Although the precision of selection of angles of polarization beam splitters is very high (e.g., in the Vienna-13 experiment [1] – private communication), nevertheless, there are errors which can lead to deviations from the expected value of the  $f = J_{\mathcal{B}}(r, \omega, \theta)$ . Therefore it is important to study the problem of the statistical stability of optimization with respect to random fluctuations of the angles. In this section we present the corresponding

theoretical considerations, the results of numerical optimization (again with the aid of Nelder-Mead method) will be presented in section 4.

Taking into account possible random fluctuations of the angles makes the question of optimization more complicated. As the result of such fluctuations, in the optimal point for the mathematical expectation the dispersion is nontrivial. In principle, one can get a large magnitude of the absolute value of the mathematical expectation, but at the same time also a large magnitude of the standard deviation. Therefore it is natural to optimize not simply the mathematical expectation given by the function  $J_{\mathcal{B}}$ , but the quantity  $J_{\mathcal{B}}/\sigma$ .

In signal processing the quantity

$$K = \mu/\sigma,$$

where  $\mu$  is average and  $\sigma$  is the standard deviation, is widely used and known as *signal/noise ratio* (SNR), see [40], [41]: This interpretation can be used even in our framework (if we interpret random fluctuations of angles as generated by a kind of noise), although we operate not with continuous signals, but with the discrete clicks of detectors. We also remark that SNR is as the reciprocal of *the coefficient of variation*,  $\sigma/\mu$ . It shows the extent of variability in relation to mean of the sample.

One of specialties of our work with SNR or the coefficient of variation is that in the standard situations they are used only for measurements with nonnegative values. In our case values are negative. However, we can simply change the sign of measurement quantity. Therefore we proceed with negative  $K$  by taking into account that statistical meaning has to be assigned to its absolute value – the reciprocal of the the relative standard deviation (RSD) which is the absolute value of the coefficient of variation,  $|\sigma/\mu|$ .

We now move to theoretical modeling of randomly fluctuating angles of polarization beam splitters. Generally any self-adjoint quantum operator  $A$  can be represented using spectral decomposition as  $A = \int_{-\infty}^{+\infty} \lambda dE_{\lambda}$ . Then its mathematical expectation value for a state  $\psi$  can be expressed as:

$$\bar{A}_{\psi} = \int_{-\infty}^{+\infty} \lambda dp_{\psi}(\lambda),$$

where  $dp_{\psi}(\lambda) = d\langle E_{\lambda} \psi, \psi \rangle$  is the probability distribution for the corresponding spectral decomposition and quantum state. Therefore for the fixed  $\psi$  quantum observable can be regarded as a classical random variable with the probability distribution  $p_{\psi}(O) = \int_O d\langle E_{\lambda} \psi, \psi \rangle, O \subset \mathbb{R}$ .

Consider the following problem. Let the observable  $A$  depend on some classical random variable  $\omega : A = A(\omega)$  corresponding to the case when angle values for the prisms positions cannot be set without an error during experiments. In this

case the spectral decomposition  $A(\omega) = \int_{-\infty}^{+\infty} \lambda dE_\lambda(\omega)$  and the density distribution function  $dp_\psi(\lambda|\omega) = d\langle E_\lambda(\omega)\psi, \psi \rangle$  also depends on this random variable. For every fixed  $\omega$  also  $\int_{-\infty}^{+\infty} dp_\psi(\lambda|\omega) = 1$  condition holds.

Let the random variable  $\omega$  be described using the Kolmogorov probability space  $(\Omega, \mathcal{F}, P)$ , where  $\Omega$  is the set of elementary events,  $\mathcal{F}$  is the  $\sigma$ -algebra of events and  $P$  is the probability measure. The mathematical expectation value of  $A(\omega)$  takes the following form:

$$\bar{A}_\psi(\omega) = \int_{\Omega} \left[ \int_{-\infty}^{+\infty} \lambda dp_\psi(\lambda|\omega) \right] dP(\omega) = \int_{\Omega} \langle A(\omega)\psi, \psi \rangle dP(\omega) = \tilde{\mathbf{E}}[\langle A(\omega)\psi, \psi \rangle],$$

where  $\tilde{\mathbf{E}}[\cdot]$  is the classical mathematical expectation.

In a similar way we obtained an expression for the dispersion  $A(\omega)$ :

$$\sigma^2(A) = \tilde{\mathbf{E}}[\langle A^2(\omega)\psi, \psi \rangle - \langle A(\omega)\psi, \psi \rangle^2].$$

## 5 The model with uniform random fluctuations of four angles of polarization beam splitters

Consider a model in which the value of each angle in the experiment is uniformly distributed at the section around the desired value. In this case mathematical expectation and dispersion values become:

$$J_{\mathcal{B}} = \frac{1}{16\delta^4} \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} \langle \mathcal{B}(x_1, x_2, x_3, x_4)\psi, \psi \rangle dx_1 dx_2 dx_3 dx_4,$$

$$\sigma^2 = \frac{1}{16\delta^4} \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} (\langle \mathcal{B}^2\psi, \psi \rangle - \langle \mathcal{B}\psi, \psi \rangle^2) dx_1 dx_2 dx_3 dx_4.$$

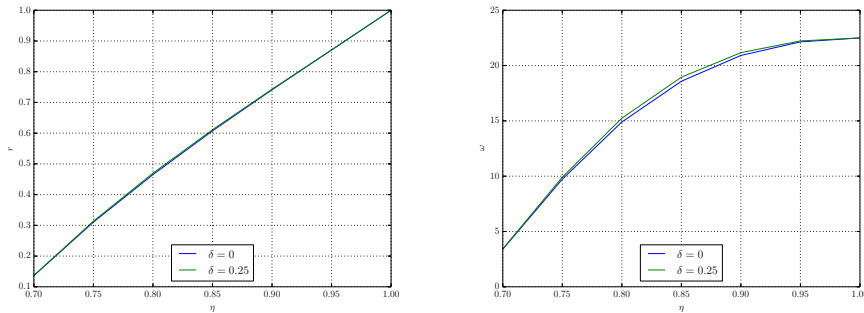


Figure 5: Optimized  $r$  and  $\omega$  values (depending on  $\eta$ ) for various values of angles

$\eta$	$\delta, ^\circ$	$r$	$\omega, ^\circ$	$\theta, ^\circ$	$J$	$J_\delta$	$\sigma_\delta$	$K = J_\delta/\sigma_\delta$
0.7	0.25	0.136389	3.40081	21.4266	-0.000453562	-0.000444565	0.00241554	-0.184044
		0.136389	3.40081	21.4266	-0.000453562	-0.000444565	0.00241554	-0.184044
		0.137124	3.42997	21.496	-0.000453514	-0.000444515	0.00241503	-0.184062
0.75	0.25	0.310518	9.73143	31.9603	-0.00615095	-0.00614082	0.00248895	-2.46724
		0.310518	9.73143	31.9603	-0.00615095	-0.00614082	0.00248895	-2.46724
		0.313658	9.91344	32.2158	-0.00614786	-0.00613773	0.00248642	-2.4685
0.8	0.25	0.465228	14.8979	37.9215	-0.02191	-0.0218985	0.002596	-8.43546
		0.465228	14.8979	37.9215	-0.02191	-0.0218985	0.002596	-8.43546
		0.469841	15.2419	38.3231	-0.0218953	-0.0218838	0.00259252	-8.44116
0.85	0.25	0.607424	18.5808	41.5341	-0.0496902	-0.0496772	0.00275221	-18.0499
		0.607424	18.5808	41.5341	-0.0496902	-0.0496772	0.00275221	-18.0499
		0.61123	18.9498	41.9292	-0.0496699	-0.0496569	0.00274992	-18.0576
0.9	0.25	0.741202	20.9153	43.6381	-0.0899078	-0.0898932	0.00296469	-30.3213
		0.741202	20.9153	43.6381	-0.0899078	-0.0898932	0.00296469	-30.3213
		0.743038	21.167	43.8961	-0.0898969	-0.0898824	0.00296395	-30.3252
0.95	0.25	0.87067	22.141	44.6958	-0.142436	-0.14242	0.00323493	-44.0258
		0.87067	22.141	44.6958	-0.142436	-0.14242	0.00323493	-44.0258
		0.871004	22.2272	44.7823	-0.142435	-0.142419	0.00323486	-44.0263
1.0	0.25	0.999997	22.5	45	-0.207107	-0.207089	0.0035626	-58.1286
		0.999997	22.5	45	-0.207107	-0.207089	0.0035626	-58.1286
		0.999999	22.4981	44.998	-0.207107	-0.207089	0.0035626	-58.1286

Table 4: Optimized parameters' values for the error in four angles separately in case of  $\delta = 0.25^\circ$



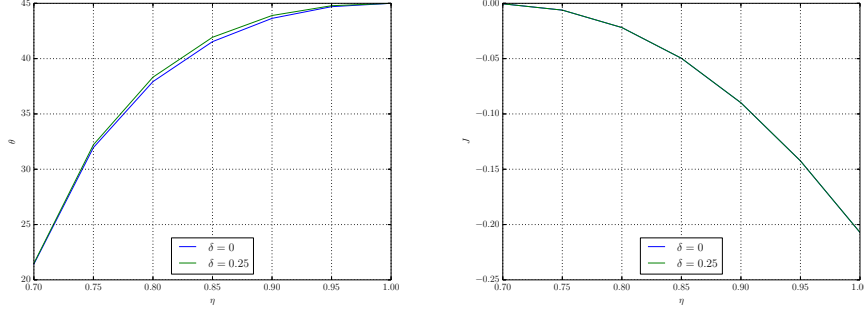


Figure 6: Optimized  $\theta$  and  $J_{\mathcal{B}}/N$  values (depending on  $\eta$ ) for various values of angles

Results of the performed optimization are shown at Fig. 5, 6 and in Table 4, its rows are also grouped in triads. It follows from the graphs that the addition of random fluctuations of the angles almost do not change optimal parameters. Therefore we can suggest that angles' values control can be reduced.

## 6 Conclusion

In this paper we analyzed the Eberhard inequality [3]. (This inequality obtained in 1993 was practically forgotten, experimentalists and theoreticians were interested mainly in the CHSH-inequality.) Our goal was to find angles of polarization beam splitters and a quantum state (which is entangled, but not maximally entangled) that allow to violate the inequality as much as possible. Required parameters were found using optimization procedure based on the Nelder-Mead optimization method [39]. We considered two models, one due to Eberhard [3], and another used for the Vienna-13 experiment, see Gustina et al. [1] and Kofler et al. [4]. In the first case we obtained values consistent with the values from the article [3]. Note that the model of Eberhard describes only the case of equal detector efficiencies. However, in real experiments, detectors' efficiencies may differ essentially. Therefore it was important to perform the study similar to [3] for detectors of different efficiencies. And such a study was done. In the second case (Vienna-13) we obtained values of parameters which differ slightly from the values used for the Vienna-13 experiment [1], [4]. Our optimal values of the angles and the state parameters provide a possibility to obtain stronger violation of the Eberhard inequality than in [1], [4]. We remark that the model of Kofler et al. [4] is asymmetric with respect to detectors' efficiencies. We explored this feature of the model and found (that is curious) that experimentalists from Vienna would be able to obtain a stronger violation of the Eberhard inequality simply by permutation of the detectors which they used for the experiment.

In both aforementioned models it was assumed that in real experiments the optimal values of the angles of polarization beam splitters (obtained as the results of optimization) can be fixed in the perfect accordance with the theoretical prediction. Although this assumption is justified up to a high degree, in real experiments there are always present errors in fixing of these angles. Such random errors have to be taken into account. This is an important part of this paper. We performed the corresponding theoretical modeling completed with numerical simulation. In this model the magnitude of the possible spread of experimental data which can be expressed using the reciprocal of the *coefficient of variation*  $\sigma_J/J$  (also known as signal/noise ratio) was studied. The obtained parameters differ from the results of Eberhard [3] and Gustina et al. [1] and Kofler et al. [4]. The simulation results can be interesting for experimenters as they allow to weaken control over the precision of orientation of the axes of polarization beam splitters.

The obtained results allow us to expect that in the experimental test of the Bell type inequality in the Eberhard form with the optimal values of physical parameters from this paper, the inequality will be significantly violated even for different detectors' efficiencies and inaccuracy in the installation angles and without the assumption of the purity of the initial state. We hope that our study may be use-

ful for experimentalists trying to perform a loophole free Bell test, i.e., trying to combine closing of the detection loophole with closing of the locality loophole.

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